

DITKIN CONDITIONS

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ABSTRACT. This paper is about the connection between certain Banach-algebraic properties of a commutative Banach algebra E with unit and the associated commutative Banach algebra $C(X, E)$ of all continuous functions from a compact Hausdorff space X into E . The properties concern Ditkin's condition and bounded relative units. We show that these properties are shared by E and $C(X, E)$. We also consider the relationship between these properties in the algebras E , B and \tilde{B} that appear in so-called admissible quadruples (X, E, B, \tilde{B}) .

1. INTRODUCTION AND PRELIMINARIES

1.1. Let A be a commutative Banach algebra with unit. The Gelfand transform $f \mapsto \hat{f}$ is a unital algebra homomorphism from A onto an algebra \hat{A} of continuous complex-valued functions on its character space $M(A)$, the set of nonzero complex-valued multiplicative linear functionals on A , equipped with the relative weak-star topology from the dual A^* . The kernel of this homomorphism is the Jacobson radical $\text{rad}(A)$, and so \hat{A} is isomorphic to A when A is semisimple. See [3] for background.

For a nonempty compact Hausdorff space X and a Banach algebra E , we let $C(X, E)$ be the space of all continuous maps from X into E . We define the *uniform norm* on $C(X, E)$ by

$$\|f\|_X = \sup_{x \in X} \|f(x)\|, \quad f \in C(X, E).$$

For $f, g \in C(X, E)$ and $\lambda \in \mathbb{C}$, the pointwise operations λf , $f + g$ and fg in $C(X, E)$ are defined as usual. It is easy to see that $C(X, E)$, equipped with the norm $\|\cdot\|_X$ is a Banach algebra. If $E = \mathbb{C}$ we get the algebra $C(X, \mathbb{C}) = C(X)$ of all continuous complex-valued functions on X . Hausner [5] showed that if E is a commutative semisimple algebra,

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then $C(X, E)$ is also semisimple, with character space homeomorphic to $X \times M(E)$.

In this paper, we consider the connection between certain Banach-algebraic properties of commutative E and of $C(X, E)$. In many cases, properties of E are inherited by $C(X, E)$. The properties concerned will be detailed shortly. We also consider inheritance of properties by certain subalgebras of $C(X, E)$ called E -valued function algebras. More specifically, we consider E -valued function algebras \tilde{B} that appear in what are called admissible quadruples (X, E, B, \tilde{B}) . We now explain this concept.

1.2. E -valued function algebras. We recall definitions from our previous paper [7]:

Definition 1.1. By an E -valued function algebra on X we mean a subalgebra $A \subset C(X, E)$, equipped with some norm that makes it complete, such that (1) A has as an element the constant function $x \mapsto 1_E$, (2) A separates points on X , i.e. given distinct points $a, b \in X$, there exists $f \in A$ such that $f(a) \neq f(b)$, and (3) the evaluation map

$$e_x : \begin{cases} A & \rightarrow E, \\ f & \mapsto f(x) \end{cases}$$

is continuous, for each $x \in X$.

Definition 1.2. By an *admissible quadruple* we mean a quadruple (X, E, B, \tilde{B}) , where

- (1) X is a compact Hausdorff space,
- (2) E is a commutative Banach algebra with unit,
- (3) $B \subset C(X)$ is a natural \mathbb{C} -valued function algebra on X ,
- (4) $\tilde{B} \subset C(X, E)$ is an E -valued function algebra on X ,
- (5) $B \cdot E \subset \tilde{B}$, and
- (6) $\{\lambda \circ f, f \in \tilde{B}, \lambda \in M(E)\} \subset B$.

One example is $(X, E, C(X), C(X, E))$. For other examples, such as Lipschitz algebras and algebras associated to E -valued polynomials, rational functions and analytic functions, see [7], and see also Subsection 1.6.

Given an admissible quadruple (X, E, B, \tilde{B}) , we define the *associated map* (also called Hausner's map)

$$\beta : \begin{cases} X \times M(E) & \rightarrow M(\tilde{B}) \\ (x, \psi) & \mapsto \psi \circ e_x. \end{cases}$$

The associated map is always injective.

Definition 1.3. We say that an admissible quadruple (X, E, B, \tilde{B}) is *natural* if the associated map β is bijective.

Each quadruple of the form $(X, E, C(X), C(X, E))$ is admissible and natural. This is a more precise statement of Hausner's lemma [5, Lemma 2].

1.3. Properties. Let A be a commutative Banach algebra with unit.

Given an element $a \in A$, the *cozero set* of a is defined as

$$\text{coz}(a) := \{\phi \in M(A) : \hat{a}(\phi) \neq 0\},$$

and the *support* $\text{supp}(a)$ as the closure of $\text{coz}(a)$ in $M(A)$.

To a closed set $S \subset M(A)$ are associated two ideals, the *kernel* of S ,

$$I_S = I_S(A) := \{a \in A : \hat{a}(S) \subset \{0\}\},$$

and the smaller ideal

$$J_S = J_S(A) := \{a \in A : \text{supp}(\hat{a}) \cap S = \emptyset\}.$$

For $\phi \in M(A)$, we abbreviate $I_\phi = I_\phi(A) := I_{\{\phi\}}$ (a maximal ideal) and $J_\phi = J_\phi(A) := J_{\{\phi\}}$.

A is said to have *bounded relative units* if, for every $\phi \in M(A)$, there exists $m_\phi > 0$ such that, for each compact subset K of $M(A) \setminus \{\phi\}$, there exists $a \in J_\phi$ with $\hat{a}(K) \subset \{1\}$ and $\|a\| \leq m_\phi$.

A satisfies *Ditkin's condition* at $\phi \in M(A)$ if $a \in \text{clos}(aJ_\phi)$ for all $a \in I_\phi$.

A is a *Ditkin algebra* if A satisfies Ditkin's condition at each $\phi \in M(A)$.

A is a *strong Ditkin algebra* if I_ϕ has a bounded approximate identity contained in J_ϕ for each $\phi \in M(A)$, i.e. there exists $m_\phi > 0$ and $u_n \in J_\phi$ ($n \in \mathbb{N}$) such that $\|u_n\| \leq m_\phi$ for each n and $\|a - au_n\| \rightarrow 0$ for each $a \in I_\phi$.

1.4. Summary of Results.

Theorem 1. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then $C(X, E)$ is Ditkin if and only if E is Ditkin.*

Theorem 2. *Let (X, E, B, \tilde{B}) be a natural admissible quadruple and suppose \tilde{B} is semisimple. Then \tilde{B} has bounded relative units if and only if both E and B have bounded relative units.*

Corollary 1.1. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then $C(X, E)$ has bounded relative units if and only if E has bounded relative units*

Corollary 1.2. *Let X be a nonempty compact Hausdorff space and E be a commutative Banach algebra with unit. Then $C(X, E)$ is a strong Ditkin algebra if and only if E is a strong Ditkin algebra.*

The “only if” direction of Theorem 1 and hence of Corollary 1.2 extends to natural admissible quadruples: (see Propositions 2.1 and Corollary 3.3), but it appears to be unknown whether the “if” direction does.

The results about quadruples apply to some so-called Tomiyama products, defined below. See Corollaries 2.2, 3.2 and 3.4

We conclude the paper with an application to automatic continuity for maps $T : C(X, E) \rightarrow C(Y, F)$. See Section 4.

1.5. Properties of Admissible Quadruples. If (X, E, B, \tilde{B}) is a natural admissible quadruple, then it is easy to see that \tilde{B} is semisimple if and only if E is semisimple.

Although E is not assumed semisimple in the definition, the quadruple concept really concerns semisimple E . The following is rather easily checked:

Proposition 1.3. *Let (X, E, B, \tilde{B}) satisfy conditions (1)-(5) of the definition. Define*

$$\hat{\tilde{B}} := \{x \mapsto \widehat{f(x)} : f \in \tilde{B}\}.$$

Then (X, E, B, \tilde{B}) is an admissible quadruple, if and only if $(X, \hat{E}, B, \hat{\tilde{B}})$ is an admissible quadruple. \square

(We emphasize that, in this proposition, \hat{E} denotes the Gelfand transform algebra with the quotient norm from $E/\text{rad}(E)$, not the supremum norm.)

Also, for semisimple E , there is sometimes symmetry in the rôles of E and B :

Definition 1.4. We say that an admissible quadruple (X, E, B, \tilde{B}) is *tight* if for each $f \in \tilde{B}$ the map

$$(1) \quad \Phi(f) : \begin{cases} M(E) & \rightarrow B \\ \lambda & \mapsto \lambda \circ f \end{cases}$$

is continuous from $M(E)$ (with the usual relative weak-star topology from E^*) to B .

Proposition 1.4. *Suppose (X, E, B, \tilde{B}) is a tight admissible quadruple, and E is semisimple. Define $\Phi(f)$ by Equation (1), for each $f \in \tilde{B}$. Then Φ is an algebra isomorphism of \tilde{B} onto a B -valued function algebra on $M(E)$, and $(M(E), B, E, \Phi(\tilde{B}))$ is an admissible quadruple.*

Proof. Since the quadruple is tight, the map Φ is a well-defined linear map from the Banach space \tilde{B} to the Banach space $C(M(E), B)$. An application of the Closed Graph theorem [2] shows that Φ is continuous. Thus $\Phi(\tilde{B})$ is a B -valued function algebra on $M(E)$. The rest is clear. \square

The following example shows that Proposition 1.4 would fail without the assumption of tightness:

Example 1.5. Let C denote the set of all continuous functions $f : [0, 1] \times [0, 2] \rightarrow \mathbb{C}$ such that the partial derivative $\frac{\partial f}{\partial y}$ exists at all points of the rectangle $R := [0, 1] \times [0, 2]$, is bounded on the whole rectangle, and is such that $\frac{\partial f}{\partial y}(x, y)$ is continuous on each vertical line, i.e. is continuous in y on $[0, 2]$ for each fixed $x \in [0, 1]$. With pointwise operations and the norm

$$\|f\|_C := \sup_R |f| + \sup_R \left| \frac{\partial f}{\partial y} \right|,$$

C is a natural function algebra on R .

Next, take $E = C^0([0, 1])$, $B = C^1([0, 2])$, and $X = [0, 1]$. Then (with pointwise operations and the usual norms) E and B are semisimple separable commutative Banach algebras, with $M(E) = [0, 1]$ and $M(B) = [0, 2]$.

Let

$$C_1 := \{F \in E^{[0,2]} : ((x, y) \mapsto F(y)(x)) \in C\},$$

$$C_2 := \{G \in B^{[0,1]} : ((x, y) \mapsto G(x)(y)) \in C\}.$$

Then C_1 is an algebra of E -valued functions on $[0, 2]$ and C_2 is an algebra of B -valued functions on $[0, 1]$, when endowed with pointwise operations. Both algebras are algebra-isomorphic to C , via obvious isomorphisms. When they are given the norms induced by these isomorphisms, (X, E, B, C_1) is an admissible quadruple, and, in the notation of Proposition 1.4, $C_2 = \Phi(C_1)$.

We claim that $(M(E), B, E, \Phi(C_1)) = ([0, 1], B, E, C_2)$ is not an admissible quadruple, because the elements of C_2 are not all *continuous* B -valued functions. To see this, we give an example of a function $f \in C$ such that

$$\begin{cases} [0, 1] & \rightarrow C^1([0, 2]), \\ x & \mapsto (y \mapsto f(x, y)), \end{cases}$$

is not continuous.

Take

$$f(x, y) = \begin{cases} 0 & , \quad 0 \leq y \leq x, \\ \frac{(y-x)^2}{2x} & , \quad x < y < 2x, \\ y - \frac{3x}{2} & , \quad 2x \leq y \leq 2, \end{cases}, 0 \leq x \leq 1.$$

Then f is continuous on R , the partial derivative $\frac{\partial f}{\partial y}$ is continuous on each vertical line and is bounded, but it is not continuous at $(0, 0)$. Moreover, the value of $\|f(x, \cdot) - f(0, \cdot)\|_B$ exceeds 1 for all $x > 0$, so it does not tend to 0 as $x \downarrow 0$.

1.6. Tensor products. Let A and B be commutative Banach algebras with unit. A *Tomiyama product of A and B* is the completion of the algebraic tensor product $A \otimes B$ with respect to some submultiplicative cross norm not less than the injective tensor product norm. See [8] and [6, Section 2.11] for background on cross norms and tensor products of Banach algebras.

Proposition 1.6. *Let C be a Tomiyama product of A and B , two commutative Banach algebras with unit. Suppose C is semisimple. Then $(M(B), A, B, C)$ is a natural admissible quadruple.*

Remark 1.1. Kaniuth shows (cf. [6, Cor. 2.11.3]) that if C is semisimple, then so are A and B . Thus, since we are mainly interested in semisimple C , we might just as well have restricted to semisimple A and B in the definition of Tomiyama product.

Tomiyama showed [9, Theorem 4] that a Tomiyama product C is automatically semisimple if both A and B are semisimple, at least one of them has the Banach approximation property and the norm is either the projective or injective product norm.

Proof of Proposition. Let $X = M(B)$.

First, we have to explain how C may be regarded as an A -valued function algebra on X (condition (4) of the definition of admissible quadruple).

By the definition of Tomiyama product, we have

$$\|f\|_{A \otimes B} \leq \|f\|_C$$

for all $f \in A \otimes B$.

Let $f \in C$. Then there is a C -norm-Cauchy sequence $f_n \in A \otimes B$ with $\|f - f_n\|_C \rightarrow 0$. Thus (f_n) is $A \check{\otimes} B$ -norm-Cauchy as well, and so converges to an element $\Psi(f) \in A \check{\otimes} B$. We have

$$\|\Psi(f)\|_{A \check{\otimes} B} = \lim \|\Psi(f_n)\|_{A \check{\otimes} B} \leq \lim \|f_n\|_C = \|f\|_C.$$

One can check that $\Psi(f)$ does not depend on which Cauchy sequence (f_n) is chosen. So we have a well-defined continuous map $\Psi : C \rightarrow A \hat{\otimes} B$, a contraction, in fact. The map Ψ is also linear, as is easily seen.

Next, we claim that Ψ is injective. Suppose $f \in C$ and $\Psi(f) = 0$. Take any sequence $f_n \in A \otimes B$ such that $f_n \rightarrow f$ in C -norm. Then $f_n \rightarrow 0$ in $A \hat{\otimes} B$ -norm.

Fix any $\chi \in M(C)$. By Tomiyama's Theorem, there exist $\lambda \in M(A)$ and $\gamma \in M(B)$ such that $\chi = \lambda \otimes \gamma$ when restricted to the algebraic tensor product $A \otimes B \subset C$. Moreover, there is a character χ' on $A \hat{\otimes} B$ that agrees with χ on $A \otimes B$.

Fix $\epsilon > 0$. Choose $n \in \mathbb{N}$ such that

$$\|f - f_n\|_C < \frac{\epsilon}{2} \text{ and } \|f_n\|_{A \hat{\otimes} B} < \frac{\epsilon}{2}.$$

Then

$$|\chi(f)| \leq |\chi(f - f_n)| + |\chi(f_n)| < \frac{\epsilon}{2} + \frac{\epsilon}{2},$$

because χ has norm 1 in C^* , $\chi(f_n) = \chi'(f_n)$, and χ' has norm 1 in $(A \otimes B)^*$. Thus $|\chi(f)| < \epsilon$ for all $\epsilon > 0$.

Thus $\chi(f) = 0$ for all $\chi \in M(C)$. Since C is semisimple, $f = 0$. Thus Ψ is injective, as claimed.

So we have a continuous injection from C into the injective tensor product $A \hat{\otimes} B$, which is a subset of $A \hat{\otimes} C(X)$, and the latter is naturally identified with $C(X, A)$, as shown by Grothendieck [8].

Conditions (1)-(3) and (5) are straightforward, and condition (6) holds because $A \otimes B$ is dense in C .

Thus (X, A, B, C) is an admissible quadruple. It is natural by Tomiyama's main result that $M(C)$ is homeomorphic to $M(A) \times M(B)$ [9, Theorem 1]. \square

Remark 1.2. The projective tensor product $A \hat{\otimes} B$ of commutative Banach algebras A and B is an example of a Tomiyama product, but it is not always semisimple if A and B are. The natural map $\Psi : A \hat{\otimes} B \rightarrow A \hat{\otimes} B$ (as in the proof above) may fail to be injective. In fact [1, Section 9] it is injective for all B if and only if A has the Banach approximation property.

Corollary 1.7. *Let C be a Tomiyama product of A and B , two semisimple commutative Banach algebras with unit. Suppose C is semisimple. Then the admissible quadruple $(M(B), A, B, C)$ is tight.*

Proof. Applying the Theorem with A and B interchanged, we conclude that $(M(A), B, A, C)$ is an admissible quadruple, so C is (isometrically isomorphic to) a B -valued function algebra on $M(A)$, i.e. $(M(B), A, B, C)$ is tight. \square

Thus we can assert that the algebra C in Example 1.5 is *not* a Tomiyama product of $C^0([0,1])$ and $C^1([0,2])$.

2. DITKIN ALGEBRAS

In this section we will prove Theorem 1. As indicated, one direction generalises to admissible quadruples:

2.1. The “only if” direction.

Proposition 2.1. *Let (X, E, B, \tilde{B}) be an admissible quadruple. Suppose \tilde{B} is Ditkin. Then E and B are Ditkin.*

Proof. Suppose \tilde{B} is Ditkin.

By Proposition 1.3, $(X, \hat{E}, B, \hat{\tilde{B}})$ is admissible, and since $\hat{\tilde{B}}$ inherits the Ditkin property, we may assume without loss in generality that E is semisimple.

To see that E is Ditkin, fix $\psi \in M(E)$, and $b \in E$ with $\psi(b) = 0$. We wish to show that there exist $b_n \in J_\psi(E)$ such that $\|b - b_n b\|_E \rightarrow 0$ as $n \uparrow \infty$.

Pick any $x_0 \in X$, and define $\phi = \beta(x_0, \psi)$. Then $\phi \in M(\tilde{B})$. Define $f(x) = b$, for all $x \in X$. Then $f \in I_\phi$, so since \tilde{B} is Ditkin we may choose $f_n \in \tilde{B}$ such that each $\hat{f}_n = 0$ near ϕ in $M(\tilde{B})$ and $\|f - f_n f\|_X \rightarrow 0$ as $n \uparrow \infty$. Take $b_n = f_n(x_0)$. Then $\|b - b_n b\|_E \rightarrow 0$. Since β is continuous, we may choose open sets $U_n \subset X$ and $V_n \subset M(E)$ such that $x_0 \in U_n$, $\psi \in V_n$ and $\hat{f}_n = 0$ on $\beta(U_n \times V_n)$. Then for $\chi \in V_n$ we have

$$\hat{b}_n(\chi) = \hat{f}_n(\beta(x_0, \chi)) = 0.$$

Thus $b_n \in J_\psi$, as desired.

If (X, E, B, \tilde{B}) were tight, we could immediately use Proposition 1.4, to deduce that $(M(E), B, E, \Phi(\tilde{B}))$ is admissible, and the isomorphic algebra $\Phi(\tilde{B})$ is Ditkin, so B is Ditkin. However we do not need to make this assumption.

Assume just that (X, E, B, \tilde{B}) is an admissible quadruple, and \tilde{B} is Ditkin. The map

$$\Psi(\lambda) : \begin{cases} \tilde{B} & \rightarrow B \\ f & \mapsto \lambda \circ f \end{cases}$$

is a well-defined algebra homomorphism, for each $\lambda \in M(E)$. By using the Closed Graph theorem, we see that $\Psi(\lambda)$ is continuous.

Now fix $a \in X$ and $g \in B$ with $g(a) = 0$. Pick any $\lambda_0 \in M(E)$ and define $\phi = \beta(a, \lambda_0)$. Then $\phi \in M(\tilde{B})$. Define $f(x) = g(x) \cdot 1_E$ for all $x \in X$. Then $f \in \tilde{B}$ and $\phi(f) = 0$, so since \tilde{B} is Ditkin we may choose $f_n \in \tilde{B}$ such that $\hat{f}_n = 0$ near ϕ in $M(\tilde{B})$ and $\|f - f_n f\|_{\tilde{B}} \rightarrow 0$. Let $g_n = \Psi(\lambda_0)(f_n)$. Then $g_n \in B$ and $g_n = 0$ near a . Since $g = \Psi(\lambda_0)(f)$ and $\Psi(\lambda_0)$ is continuous, we have

$$\|g - g_n g\|_B \rightarrow 0.$$

Thus B is Ditkin. \square

Applying Proposition 1.6, we have:

Corollary 2.2. *Let A and B be semisimple commutative Banach algebras with unit, and let C be a semisimple Tomiyama product of A and B . Suppose C is Ditkin. Then so are A and B .* \square

2.2. Converse direction. Turning to the other direction, we restrict to the special quadruple $(X, E, C(X), C(X, E))$:

Proposition 2.3. *Let X be a compact Hausdorff space and let E be a commutative Banach algebra with unit. Suppose E is Ditkin. Then $C(X, E)$ is Ditkin.*

Proof. Fix $\phi \in M(C(X, E))$, and $f \in I_\phi$. Let $\epsilon > 0$ be given.

Choose $\psi \in M(E)$ and $x_0 \in X$ such that $\phi = \beta(x_0, \psi)$.

Let $a = f(x_0)$. Then $\psi(a) = 0$. Since E is Ditkin, we may choose $b \in J_\psi(E)$ such that $\|a - ba\|_E < \epsilon$. Let

$$U = \{x \in X : \|f(x)b - f(x)\|_E < \epsilon\}.$$

Then U is an open neighbourhood of x_0 . Thus, by Urysohn's Lemma, we may choose $h \in J_{x_0}(C(X))$ with $h = 1$ off U and $0 \leq h \leq 1$ on X .

Then for each $x \in X$, we have

$$\|(1 - h(x))(bf(x) - f(x))\|_E < \epsilon.$$

Thus $\|(1 - h)(bf - f)\|_X < \epsilon$. Now

$$f + (1 - h)(bf - f) = f(h \cdot 1_E + b - hb) \in fJ_\phi$$

(since $h \cdot 1_E + b - hb = 0$ on $\beta^{-1}(h^{-1}(0) \cap b^{-1}(0))$), so the distance from f to fJ_ϕ in $C(X, E)$ norm is less than ϵ .

The result follows. \square

Proof of Theorem 1. Apply Proposition 2.1 (with $B = C(X)$ and $\tilde{B} = C(X, E)$) and Proposition 2.3. \square

3. BOUNDED RELATIVE UNITS

3.1. **Proof of Theorem 2.** Note the following:

Lemma 3.1. *If A is a commutative Banach algebra with identity, then the following are equivalent:*

- (1) *A has bounded relative units.*
- (2) *For each $\phi \in M(A)$, there exists a constant $c_\phi > 0$ with the following property: for every closed subset K of $M(A)$ with $\phi \notin K$, there exists $x \in A$ such that $\|x\| \leq c_\phi$, $\hat{x} = 0$ on K and $\hat{x} = 1$ on some neighbourhood of ϕ .*

Proof. Let e denote the identity of A .

Suppose (1) holds. Fix $\phi \in M(A)$, and let $m_\phi > 0$ be chosen as in the definition of bounded relative units. Take $c_\phi = m_\phi + \|e\|$. Let $K \subset M(A)$ be compact, with $\phi \notin K$. We may choose $a \in J_\phi$ such that $\hat{a}(K) \subset \{1\}$ and $\|a\| \leq m_\phi$. Taking $x = e - a$ we have $\|x\| \leq c_\phi$, $\hat{x} = 0$ on K and $\hat{x} = 1$ near ϕ . Thus (2) holds.

The other direction is similar. \square

This shows, in particular, that a unital commutative Banach algebra with bounded relative units is regular.

Proof of Theorem 2. For the “only if” direction, suppose (X, E, B, \tilde{B}) is an admissible quadruple, and \tilde{B} has bounded relative units. Then we have to show that E and B have bounded relative units.

First, consider E , and fix $\psi_0 \in M(E)$. Fix any $x_0 \in X$. Since the evaluation map $f \mapsto f(x_0)$ is continuous from $\tilde{B} \rightarrow E$, there exists $\kappa > 0$ such that $\|f(x_0)\|_E \leq \kappa \|f\|_{\tilde{B}}$ for all $f \in \tilde{B}$.

Define $\phi := \beta(x_0, \psi_0) \in M(\tilde{B})$. By assumption, there exists $m > 0$ such that for each open neighbourhood W of ϕ there exists $f \in J_\phi$ such that $\hat{f} = 1$ off W and $\|f\|_{\tilde{B}} \leq m$. Let $F \subset M(E) \setminus \{\psi_0\}$ be a compact subset. Define $L := \{\beta(x_0, \chi) : \chi \in F\}$. It is clear that L is a compact subset of $M(C(X, E)) \setminus \{\phi\}$. We may choose $f \in J_\phi$ such that $\hat{f}(L) \subset \{1\}$ and $\|f\|_{\tilde{B}} \leq m$. Define $b := f(x_0)$. Then $b \in E$, $\hat{b}(F) \subset \{1\}$ and $\|b\|_E \leq \kappa \|f\|_{\tilde{B}} \leq \kappa m$.

Thus E has bounded relative units.

Now consider B , and fix $x_0 \in X = M(B)$. Fix any $\psi_0 \in M(E)$. As noted in the proof of Proposition 2.1, the map $f \mapsto \psi_0 \circ f$ is continuous from $\tilde{B} \rightarrow B$, so there exists $\kappa > 0$ such that $\|\psi_0 \circ f\|_B \leq \kappa \|f\|_{\tilde{B}}$. So defining $\phi := \beta(x_0, \psi_0) \in M(\tilde{B})$, we may proceed in a very similar way to the above, to deduce that B has bounded relative units.

For the ‘if’ direction, the key observation (for which the authors would like to thank the referee) uses the classical automatic continuity

theorem of Shilov [3, Theorem 2.3.3, p. 192] that each homomorphism from a Banach algebra into a semisimple commutative Banach algebra is necessarily continuous. We may apply this to the two homomorphisms

$$\left\{ \begin{array}{ccc} E & \rightarrow & \tilde{B} \\ a & \mapsto & 1_X \cdot a \end{array} \right\} \text{ and } \left\{ \begin{array}{ccc} B & \rightarrow & \tilde{B} \\ f & \mapsto & f \cdot e \end{array} \right\}$$

where e is the identity of E , and deduce that there exist constants $\alpha > 0$ and $\gamma > 0$ such that $\|1_X \cdot a\|_{\tilde{B}} \leq \alpha \|a\|_E$ for all $a \in E$ and $\|f \cdot e\|_{\tilde{B}} \leq \gamma \|f\|_B$ for all $f \in B$.

Now every $\phi \in M(\tilde{B})$ is of the form $\psi \circ e_x$ for some $\psi \in M(E)$ and some $x \in X$. Let c_x and c_ψ be constants as guaranteed by the assumption that B and E have bounded relative units. Since $M(\tilde{B})$ is homeomorphic to $X \times M(E) = M(B) \times M(E)$, given a closed subset K of $M(\tilde{B})$ such that $\phi \notin K$, we may find closed subsets $C \subset X$ and $D \subset M(E)$ such that $K \subset (C \times M(E)) \cup (X \times D)$ and $x \notin C$ and $\psi \notin D$. Then, by hypothesis, there exist

- $f \in B$ such that $\|f\|_B \leq c_x$, $f = 0$ on C and $f = 1$ on a neighbourhood of x ;
- $a \in E$ such that $\|a\|_E \leq c_\psi$, $\hat{a} = 1$ on D and $\hat{a} = 1$ on a neighbourhood of ψ .

Then the element $f \times a$ of \tilde{B} satisfies $\widehat{f \cdot a} = 0$ on $(C \times M(E)) \cup (X \times D)$ and $\widehat{f \cdot a} = 1$ in a neighbourhood of ϕ . Moreover,

$$\|f \cdot a\|_{\tilde{B}} \leq \|f \cdot e\|_{\tilde{B}} \cdot \|1_X \cdot a\|_{\tilde{B}} \leq \alpha \gamma \|a\|_E \|f\|_B \leq \alpha \gamma c_x c_\psi.$$

Thus \tilde{B} has bounded relative units. □

Corollary 3.2. *Let A and B be semisimple commutative Banach algebras with unit. Suppose that C is a semisimple Tomiyama product of A and B . Then C has bounded relative units if and only if both A and B have bounded relative units.* □

3.2. Proof of Corollaries.

Proof of Corollary 1.1. This is immediate from Theorem 2, because E has bounded relative units if and only if \hat{E} does, and $C(X, \hat{E})$ is semisimple, so the theorem applies to the quadruple $(X, \hat{E}, C(X), C(X, \hat{E}))$, and tells us that $C(X, \hat{E})$ has bounded relative units if and only if $\widehat{C(X, E)}$ does. But $C(X, \hat{E})$ is isometrically algebra-isomorphic to $\widehat{C(X, E)}$, so $C(X, E)$ has bounded relative units if and only if E does. □

Proof of Corollary 1.2. This follows from Theorems 1 and 2, since an algebra is a strong Ditkin algebra if and only if it is Ditkin and has bounded relative units [3, pp.417-8]. \square

As indicated earlier, one direction of Corollary 1.2 generalises to natural admissible quadruples:

Corollary 3.3. *Let (X, E, B, \tilde{B}) be an admissible quadruple. Suppose \tilde{B} is strong Ditkin. Then E and B are strong Ditkin.*

Proof. This follows from Proposition 2.1 and Theorem 2. \square

Corollary 3.4. *Let A and B be semisimple commutative Banach algebras with unit, and let C be a semisimple Tomiyama product of A and B . Suppose C is strong Ditkin. Then so are A and B .* \square

4. SEPARATING BIJECTIONS

Definition 4.1. Let A and B be two semisimple commutative Banach algebras with identity. A linear map $T : A \rightarrow B$ is said to be *separating* or *disjointness preserving* if $\text{coz}(Tf) \cap \text{coz}(Tg) = \emptyset$ whenever $f, g \in A$ satisfy $\text{coz}(f) \cap \text{coz}(g) = \emptyset$. Moreover, T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, a map $T : A \rightarrow B$ is separating if it is linear and $Tf \cdot Tg \equiv 0$, whenever $f, g \in A$ satisfy $f \cdot g \equiv 0$. As an application of Theorem 1, we obtain:

Theorem 3. *Let X, Y be two compact Hausdorff spaces and E, F be unital commutative semisimple Banach algebras which are Ditkin algebras and $T : C(X, E) \rightarrow C(Y, F)$ be a separating linear bijection, then*

- (i) T is continuous,
- (ii) T^{-1} is separating, and
- (iii) $X \times M(E)$ and $Y \times M(F)$ are homeomorphic.

Proof. Use [4, Theorem 1] and Theorem 1. \square

Remark 4.1. The results of this paper may be extended to semisimple commutative Banach algebras without identity by the device of adjoining a unit. We have confined attention to algebras with unit, to avoid clutter.

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